Proc. Indian Acad. Sci. (Math. Sci.) Vol. 115, No. 3, August 2003, pp. 339–345.
Printed in India

Fixed point theory for composite maps on almost dominating extension spaces

RAVI P AGARWAL[1], JONG KYU KIM[2] and DONAL O'REGAN[3]

[1]Department of Mathematical Sciences, Florida Institute of Technology, Melbourne, Florida 32901-6975, USA
[2]Department of Mathematics, Kyungnam University, Masan, Kyungnam 631-701, Korea
[3]Department of Mathematics, National University of Ireland, Galway, Ireland
E-mail: agarwal@fit.edu

MS received 4 February 2004; revised 1 February 2005

Abstract. New fixed point results are presented for $\mathscr{U}_c^\kappa(X,X)$ maps in extension type spaces.

Keywords. Composite maps; fixed points; extension spaces.

1. Introduction

In this paper we present new fixed point results in extension type spaces. In particular, we present results for compact upper semicontinuous $\mathscr{U}_c^\kappa(X,X)$ maps where X is almost ES dominating or more generally almost Schauder admissible dominating (these concepts will be defined in §2). The results in this paper improve those in the literature (see [3,4,5,9,11] and the references therein). A continuation theorem is also discussed when the maps are between topological vector spaces.

For the remainder of this section we present some definitions and known results which will be needed throughout this paper. Suppose X and Y are topological spaces. Given a class $\mathscr{X}$ of maps, $\mathscr{X}(X,Y)$ denotes the set of maps $F\colon X \to 2^Y$ (nonempty subsets of Y) belonging to $\mathscr{X}$, and $\mathscr{X}_c$ the set of finite compositions of maps in $\mathscr{X}$. We let

$$\mathscr{F}(\mathscr{X}) = \{Z\colon \text{Fix } F \neq \emptyset \quad \text{for all } F \in \mathscr{X}(Z,Z)\},$$

where Fix F denotes the set of fixed points of F.

$\mathscr{U}$ will be the class of maps [11] with the following properties:

(i) $\mathscr{U}$ contains the class $\mathscr{C}$ of single valued continuous functions;
(ii) each $F \in \mathscr{U}_c$ is upper semicontinuous and compact valued; and
(iii) $B^n \in \mathscr{F}(\mathscr{U}_c)$ for all $n \in \{1,2,\dots\}$; here $B^n = \{x \in \mathbf{R}^n\colon \|x\| \leq 1\}$.

$\mathscr{U}_c^\kappa(X,Y)$ will consist of all maps $F\colon X \to 2^Y$ such that for each F and each nonempty compact subset K of X there exists a map $G \in \mathscr{U}_c(K,Y)$ such that $G(x) \subseteq F(x)$ for all $x \in K$.

Recall [9] that $\mathscr{U}_c^\kappa$ is closed under compositions. We also discuss special examples of $\mathscr{U}_c^\kappa$ maps. Let X and Y be subsets of Hausdorff topological vector spaces E_1 and

E_2 respectively. We will consider maps $F\colon X \to K(Y)$; here $K(Y)$ denotes the family of nonempty compact subsets of Y. We say $F\colon X \to K(Y)$ is *Kakutani* if F is upper semicontinuous with convex values. A nonempty topological space is said to be acyclic if all its reduced Čech homology groups over the rationals are trivial. Now $F\colon X \to K(Y)$ is *acyclic* if F is upper semicontinuous with acyclic values. $F\colon X \to K(Y)$ is said to be an *O'Neill* map if F is continuous and if the values of F consist of one or m acyclic components (here m is fixed).

Given two open neighborhoods U and V of the origins in E_1 and E_2 repectively, a (U,V)-approximate continuous selection of $F\colon X \to K(Y)$ is a continuous function $s\colon X \to Y$ satisfying

$$s(x) \in (F\,[(x+U)\cap X]+V)\cap Y \quad \text{for every } x \in X.$$

We say $F\colon X \to K(Y)$ is *approximable* if it is upper semicontinuous and if its restriction $F|_K$ to any compact subset K of X admits a (U,V)-approximate continuous selection for every open neighborhood U and V of the origins in E_1 and E_2 respectively.

For our next definition let X and Y be metric spaces. A continuous single-valued map $p\colon Y \to X$ is called a Vietoris map if the following two conditions are satisfied:

(i) for each $x \in X$, the set $p^{-1}(x)$ is acyclic;
(ii) p is a proper map i.e. for every compact $A \subseteq X$ we have that $p^{-1}(A)$ is compact.

DEFINITION 1.1.

A multifunction $\phi\colon X \to K(Y)$ is *admissible* (strongly) in the sense of Gorniewicz, if $\phi\colon X \to K(Y)$ is upper semicontinuous, and if there exists a metric space Z and two continuous maps $p\colon Z \to X$ and $q\colon Z \to Y$ such that

(i) p is a Vietoris map, and
(ii) $\phi(x) = q(p^{-1}(x))$ for any $x \in X$.

It should be noted that ϕ upper semicontinuous is redundant in Definition 1.1. Notice the Kakutani maps, the acyclic maps, the O'Neill maps, the approximable maps and the maps admissible in the sense of Gorniewicz are examples of $\mathscr{U}_c^\kappa$ maps.

2. Fixed point theory

We begin with a result which extends results in the literature [3,11] for $\mathscr{U}_c^\kappa$ maps. In this paper by a space we mean a Hausdorff topological space. Let Q be a class of topological spaces. A space Y is an *extension space* for Q (written $Y \in ES(Q)$) if $\forall X \in Q$, $\forall K \subseteq X$ closed in X, any continuous function $f_0\colon K \to Y$ extends to a continuous function $f\colon X \to Y$. The following result was established in [4].

Theorem 2.1. *Let $X \in ES(compact)$ and $F \in \mathscr{U}_c^\kappa(X,X)$ a compact map. Then F has a fixed point.*

We begin with topological vector spaces, so let E be a Hausdorff topological vector space and $X \subseteq E$. Let V be a neighborhood of the origin 0 in E. X is said to be *ES*

V-dominated if there exists a space $X_V \in ES(\text{compact})$ and two continuous functions $r_V : X_V \to X$, $s_V : X \to X_V$ such that $x - r_V\, s_V(x) \in V$ for all $x \in X$. X is said to be *almost ES dominated* if X is ES V-dominated for every neighborhood V of the origin 0 in E.

Any space that is dominated by a normed space (or more generally a complete metric topological space admissible in the sense of Klee) is almost ES dominated.

Theorem 2.2. *Let X be a subset of a Hausdorff topological vector space E. Also assume X is almost ES dominated and that $F \in \mathscr{U}_c^{\kappa}(X,X)$ is a compact closed map. Then F has a fixed point.*

Proof. Let $\mathscr{N}$ be a fundamental system of neighborhoods of the origin 0 in E and $V \in \mathscr{N}$. Let $K = \overline{F(X)}$. Now there exists $x_V \in ES(\text{compact})$ and two continuous functions $r_V : X_V \to X$, $s_V : X \to X_V$ such that $x - r_V s_V(x) \in V$ for all $x \in X$.

Let us look at the map $G_V = s_V F r_V$. Since $\mathscr{U}_c^{\kappa}$ is closed under compositions we have that $G_V \in \mathscr{U}_c^{\kappa}(X_V, X_V)$ is a compact map. Now Theorem 2.1 guarantees that there exists $z_V \in X_V$ with $z_V \in s_V F r_V(z_V)$ i.e. $z_V = s_V(x_V)$ for some $x_V \in F r_V(z_V)$. In particular $r_V(z_V) = r_V s_V(x_V)$ so $x_V \in F r_V(z_V) = F r_V s_V(x_V)$.

Let $y_V = r_V s_V(x_V)$. Then $x_V \in F y_V$ and since $x - r_V s_V(x) \in V$ for all $x \in X$ we have $x_V - y_V \in V$. Now since $K = \overline{F(X)}$ is compact we may assume without loss of generality that there exists a x with $x_V \to x$. Also since $x_V - y_V \in V$ we have $y_V \to x$. These together with the facts that F is closed and $x_V \in F(y_V)$ implies $x \in F(x)$. □

Let Q be a class of topological spaces and let Y be a subset of a Hausdorff topological vector space. Y is a *Klee approximate extension space* for Q (written $Y \in KAES(Q)$) if for all neighborhoods V of 0, $\forall X \in Q$, $\forall K \subseteq X$ closed in X, and any continuous function $f_0 : K \to Y$, there exists a continuous function $f_V : X \to Y$ with $f_V(x) - f_0(x) \in V$ for all $x \in K$.

Let E be a Hausdorff topological vector space and $X \subseteq E$. We say X is *KAES admissible* if for every compact subset K of X and every neighborhood V of 0 there exists a continuous function $h_V : K \to X$ such that

(i) $x - h_V(x) \in V$ for all $x \in K$ and
(ii) $h_V(K)$ is contained in a subset C of X with $C \in KAES(\text{compact})$.

The following result was established in [3].

Theorem 2.3. *Let X be a subset of a Hausdorff topological vector space E. Also assume X is KAES admissible and that $F \in \mathscr{U}_c^{\kappa}(X,X)$ is a upper semicontinuous compact map with closed values. Then F has a fixed point.*

Let E be a Hausdorff topological vector space and $X \subseteq E$. Let V be a neighborhood of the origin 0 in E. X is said to be *KAES V-dominated* if there exists a KAES admissible space X_V and two continuous functions $r_V : X_V \to X$, $s_V : X \to X_V$ such that $x - r_V\, s_V(x) \in V$ for all $x \in X$. X is said to be *almost KAES dominated* if X is KAES V-dominated for every neighborhood V of the origin 0 in E.

Essentially the same reasoning as in Theorem 2.2 (except here we use Theorem 2.3) yields the following result.

Theorem 2.4. *Let X be a subset of a Hausdorff topological vector space E. Also assume X is almost KAES dominated and that $F \in \mathscr{U}_c^{\kappa}(X,X)$ is a compact upper semicontinuous map with closed values. Then F has a fixed point.*

Let X be a subset of a Hausdorff topological vector space. Then X is said to be *q-almost KAES dominated* if any nonempty compact convex subset Ω of X is almost KAES dominated.

Theorem 2.5. *Let X be a closed convex q-almost KAES dominated subset of a Hausdorff topological vector space with $x_0 \in X$. Suppose $F \in \mathscr{U}_c^{\kappa}(X,X)$ is a upper semicontinuous map with closed values and assume the following condition holds*:

$$\textit{If } A \subseteq X \textit{ with } A = \overline{co}(\{x_0\} \cup F(A)), \textit{ then } A \textit{ is compact.} \tag{2.1}$$

Then F has a fixed point.

Proof. Consider $\mathscr{F}$ the family of all closed, convex subsets C of X with $x_0 \in C$ and $F(x) \subseteq C$ for all $x \in C$ and let $C_0 = \cap_{C \in \mathscr{F}} C$. Now [2] guarantees that

$$C_0 = \overline{co}\,(\{x_0\} \cup F(C_0)). \tag{2.2}$$

Now (2.1) guarantees that C_0 is compact and (2.2) implies $F(C_0) \subseteq C_0$. Also C_0 is almost KAES dominated. In addition $F|_{C_0} \in \mathscr{U}_c^{\kappa}(C_0, C_0)$ is a compact upper semicontinuous map with closed values so Theorem 2.4 guarantees that there exists $x_0 \in C$ with $x_0 \in Fx_0$. □

Next we present a continuation theorem. Let E be a Hausdorff topological vector space, C a closed convex subset of E, U an open subset of C and $0 \in U$. We would like to consider maps $F\colon \overline{U} \to K(C)$ which are upper semicontinuous and either (a) Kakutani; (b) acyclic; (c) O'Neill; (d) approximable; or (e) admissible (strongly) in the sense of Gorniewicz. Here $\overline{U}$ denotes the closure of U in C.

To take care of all the above maps (and even more general types) we define as follows.

DEFINITION 2.1.

$F \in GA(\overline{U}, C)$ if $F\colon \overline{U} \to K(C)$ is upper semicontinuous and satisfies condition (C).

We assume condition (C) as:

$$\begin{cases} \text{for any map } F \in GA(\overline{U}, C) \text{ and any continuous single-valued} \\ \text{map } \mu\colon \overline{U} \to [0,1] \text{ we have that } \mu F \text{ satisfies condition } (C). \end{cases} \tag{2.3}$$

Certainly if condition (C) means (a), (b), (c), (d) or (e) above, then (2.3) holds.

We assume the map $F\colon \overline{U} \to K(C)$ satisfies one of the following conditions:

(H1) $F\colon \overline{U} \to K(C)$ is compact;
(H2) if $D \subseteq \overline{U}$ and $D \subseteq \overline{co}\,(\{0\} \cup F(D))$ then $\overline{D}$ is compact.

Fix $i \in \{1, 2\}$.

DEFINITION 2.2.

We say $F \in GA^i(\overline{U}, C)$ if $F \in GA(\overline{U}, C)$ satisfies (Hi).

DEFINITION 2.3.

We say $F \in GA^i_{\partial U}(\overline{U}, C)$ if $F \in GA^i(\overline{U}, C)$ with $x \notin F(x)$ for $x \in \partial U$; here ∂U denotes the boundary of U in C.

DEFINITION 2.4.

A map $F \in GA^i_{\partial U}(\overline{U},C)$ is essential in $GA^i_{\partial U}(\overline{U},C)$ if for every $G \in GA^i_{\partial U}(\overline{U},C)$ with $G|_{\partial U} = F|_{\partial U}$ there exists $x \in U$ with $x \in G(x)$.

The following result was established in [1].

Theorem 2.6. *Fix $i \in \{1,2\}$. Let E be a Hausdorff topological vector space, C a closed convex subset of E, U an open subset of C, $0 \in U$ and assume (2.3) holds. Suppose $F \in GA^i(\overline{U},C)$ and assume the following two conditions are satisfied*:

$$\textit{the zero map is essential in } GA^i_{\partial U}(\overline{U},C) \tag{2.4}$$

and

$$x \notin \lambda F x \quad \textit{for } x \in \partial U \quad \textit{and} \quad \lambda \in (0,1]. \tag{2.5}$$

Then F is essential in $GA^i_{\partial U}(\overline{U},C)$.

Next we discuss Assumption (2.4). The results presented were motivated partly by [10].

Example 2.1. Let $i = 1$. Suppose condition (C) in Definition 2.1 means $F\colon \overline{U} \to K(C)$ is either (a) Kakutani; (b) acyclic; (c) O'Neill; or (d) approximable, and assume C is almost KAES dominated. Then (2.3) and (2.4) hold.

To see (2.4) let $\theta \in GA^1_{\partial U}(\overline{U},C)$ with $\theta|_{\partial U} = \{0\}$. We must show that there exists $x \in U$ with $x \in \theta(x)$. Let

$$J(x) = \begin{cases} \theta(x), & x \in \overline{U} \\ \{0\}, & \text{otherwise.} \end{cases}$$

It is easy to see that $J\colon C \to K(C)$ is (a) Kakutani if θ is Kakutani, (b) acyclic if θ is acyclic, (c) O'Neill if θ is O'Neill or (d) approximable [10] if θ is approximable. Clearly $J\colon C \to K(C)$ is compact. Now Theorem 2.4 guarantees that there exists $x \in C$ with $x \in J(x)$. If $x \notin U$ we have $x \in J(x) = \{0\}$, which is a contradiction since $0 \in U$. Thus $x \in U$ so $x \in J(x) = \theta(x)$.

Example 2.2. Let $i = 2$. Suppose condition (C) in Definition 2.1 means $F\colon \overline{U} \to K(C)$ is either (a) Kakutani; (b) acyclic; (c)'O'Neill; or (d) approximable, and assume C is closed, convex and q-almost KAES dominated. Also suppose

$$\overline{co}\,(K) \text{ is compact for any compact subset } K \text{ of } C. \tag{2.6}$$

Then (2.3) and (2.4) hold.

To see (2.4) let $\theta \in GA^2_{\partial U}(\overline{U},C)$ with $\theta|_{\partial U} = \{0\}$ and let J be as in Example 2.1. Again $J\colon C \to K(C)$ is (a) Kakutani if θ is Kakutani, (b) acyclic if θ is acyclic, (c) O'Neill if θ is O'Neill, or (d) approximable if θ is approximable. Next we show that J satisfies (2.1) (with F replaced by J and x_0 by 0). To see this let $D \subseteq C$ with $D = \overline{co}\,(\{0\} \cup J(D))$. Then

$$D \subseteq \overline{co}\,(\{0\} \cup \theta(D \cap U)), \tag{2.7}$$

and so

$$D \cap U \subseteq \overline{co}\,(\{0\} \cup \theta(D \cap U)).$$

Since $\theta \in GA^2(\overline{U},C)$ we have that $\overline{D\cap U}$ is compact. In addition θ upper semicontinuous guarantees that $\theta(\overline{D\cap U})$ is compact, and this together with (2.6) implies that $\overline{co}\left(\{0\}\cup\theta(\overline{D\cap U})\right)$ is compact. Now (2.7) implies $\overline{D}(=D)$ is compact, so (2.1) holds. Theorem 2.5 guarantees that there exists $x\in C$ with $x\in J(x)$. As in Example 2.1 we have $x\in U$ so $x\in J(x)=\theta(x)$.

Next we extend the results of this section to Hausdorff topological spaces. First we gather together some preliminaries. For a subset K of a topological space X, we denote by $\mathrm{Cov}_X(K)$ the set of all coverings of K by open sets of X (usually we write $\mathrm{Cov}(K)=\mathrm{Cov}_X(K)$). Given a map $F\colon X\to 2^X$ and $\alpha\in\mathrm{Cov}(X)$, a point $x\in X$ is said to be an α-fixed point of F if there exists a member $U\in\alpha$ such that $x\in U$ and $F(x)\cap U\neq\emptyset$. Given two maps $F,G\colon X\to 2^Y$ and $\alpha\in\mathrm{Cov}(Y)$, F and G are said to be α-close if for any $x\in X$ there exists $U_x\in\alpha$, $y\in F(x)\cap U_x$ and $w\in G(x)\cap U_x$.

The following result can be found in [5].

Theorem 2.7. *Let X be a regular topological space and $F\colon X\to 2^X$ an upper semicontinuous map with closed values. Suppose there exists a cofinal family of coverings $\theta\subseteq\mathrm{Cov}_X(\overline{F(X)})$ such that F has an α-fixed point for every $\alpha\in\theta$. Then F has a fixed point.*

Remark 2.1. From Theorem 2.7 in proving the existence of fixed points in uniform spaces for upper semicontinuous compact maps with closed values it suffices (p. 298 of [6]) to prove the existence of approximate fixed points (since open covers of a compact set A admit refinements of the form $\{U[x]\colon x\in A\}$ where U is a member of the uniformity (p. 199 of [8]) and such refinements form a cofinal family of open covers). Note also that uniform spaces are regular (in fact completely regular) (p. 431 of [7]). Note in Theorem 2.7 if F is compact valued then the assumption that X is regular can be removed. For convenience, in this paper we will apply Theorem 2.7 only when the space is uniform.

Let X be a Hausdorff topological space and let $\alpha\in\mathrm{Cov}(X)$. X is said to be *ES α-dominated* if there exists a space $X_\alpha\in ES(\text{compact})$ and two continuous functions $r_\alpha\colon X_\alpha\to X$, $s_\alpha\colon X\to X_\alpha$ such that $r_\alpha s_\alpha\colon X\to X$ and $i\colon X\to X$ are α-close. X is said to be *almost ES dominated* if X is ES α-dominated for each $\alpha\in\mathrm{Cov}(X)$.

Theorem 2.8. *Let X be a uniform space and let X be almost ES dominated. Also suppose $F\in\mathscr{U}_c^\kappa(X,X)$ is a compact upper semicontinuous map with closed values. Then F has a fixed point.*

Proof. Let $K=\overline{F(X)}$ and $\alpha\in\mathrm{Cov}_X(K)$. Now there exists $X_\alpha\in ES(\text{compact})$ and two continuous functions $r_\alpha\colon X_\alpha\to X$, $s_\alpha\colon X\to X_\alpha$ such that $r_\alpha s_\alpha$ and i are α-close. Notice $s_\alpha F r_\alpha\in\mathscr{U}_c^\kappa(X_\alpha,X_\alpha)$ so Theorem 2.1 guarantees that there exists $z_\alpha\in X_\alpha$ with $z_\alpha\in s_\alpha F r_\alpha(z_\alpha)$ i.e. $z_\alpha=s_\alpha(x_\alpha)$ for some $x_\alpha\in Fr_\alpha(z_\alpha)$. Notice $x_\alpha\in Fr_\alpha s_\alpha(x_\alpha)$ also. Let $y_\alpha=r_\alpha s_\alpha(x_\alpha)$ and note $x_\alpha\in Fy_\alpha$. Since $r_\alpha s_\alpha$ and i are α-close there exists $U_\alpha\in\alpha$ with $r_\alpha s_\alpha(x_\alpha)\in U_\alpha$ and $x_\alpha\in U_\alpha$ i.e. $y_\alpha\in U_\alpha$ and $x_\alpha\in U_\alpha$. As a result $y_\alpha\in U_\alpha$ and $F(y_\alpha)\cap U_\alpha\neq\emptyset$ since $x_\alpha\in F(y_\alpha)$ and $x_\alpha\in U_\alpha$. Thus F has an α-fixed point. The result now follows from Theorem 2.7 (with Remark 2.1). □

Let Q be a class of topological spaces and Y a subset of a Hausdorff topological space. A space Y is an *approximate extension space* for Q (written $Y\in AES(Q)$) if $\forall\,\alpha\in\mathrm{Cov}(Y)$,

$\forall X \in Q$, $\forall K \subseteq X$ closed in X, and any continuous function $f_0\colon K \to Y$, there exists a continuous function $f\colon X \to Y$ such that $f|_K$ is α-close to f_0.

Let X be a uniform space. Then X is *Schauder admissible* if for every compact subset K of X and every covering $\alpha \in \mathrm{Cov}_X(K)$, there exists a continuous function (called the Schauder projection) $\pi_\alpha\colon K \to X$ such that

(i) π_α and $i\colon K \hookrightarrow X$ are α-close;
(ii) $\pi_\alpha(K)$ is contained in a subset $C \subseteq X$ with $C \in AES(\text{compact})$.

The following result was established in [9].

Theorem 2.9. *Let X be a uniform space and assume X is Schauder admissible. Suppose $F \in \mathscr{U}_c^\kappa(X,X)$ is a compact upper semicontinuous map with closed values. Then F has a fixed point.*

Let X be a Hausdorff topological space and let $\alpha \in \mathrm{Cov}(X)$. X is said to be *Schauder admissible α-dominated* if there exists a Schauder admissible space X_α and two continuous functions $r_\alpha\colon X_\alpha \to X$, $s_\alpha\colon X \to X_\alpha$ such that $r_\alpha s_\alpha\colon X \to X$ and $i\colon X \to X$ are α-close. X is said to be *almost Schauder admissible dominated* if X is Schauder admissible α-dominated for each $\alpha \in \mathrm{Cov}(X)$.

Essentially the same reasoning as in Theorem 2.8 (except here we use Theorem 2.9) yields the following result.

Theorem 2.10. *Let X be a uniform space and let X be almost Schauder admissible dominated. Also suppose $F \in \mathscr{U}_c^\kappa(X,X)$ is a compact upper semicontinuous map with closed values. Then F has a fixed point.*

References

[1] Agarwal R P and O'Regan D, Homotopy and Leray–Schauder principles for multimaps, *Nonlinear Analysis Forum* **7** (2002) 103–111
[2] Agarwal R P and O'Regan D, Collectively fixed point theorems, *Nonlinear Analysis Forum* **7** (2002) 167–179
[3] Agarwal R P and O'Regan D, Fixed point theory for multimaps defined on admissible subsets of topological vector spaces, *Comm. Math. (Prace Mat.)* **44** (2004) 1–10
[4] Agarwal R P, O'Regan D and Park S, Fixed point theory for multimaps in extension type spaces, *J. Korean Math. Soc.* **39** (2002) 579–591
[5] Ben-El-Mechaiekh H, The coincidence problem for compositions of set valued maps, *Bull. Austral. Math. Soc.* **41** (1990) 421–434
[6] Ben-El-Mechaiekh H, Spaces and maps approximation and fixed points, *J. Comput. Appl. Math.* **113** (2000) 283–308
[7] Engelking R, General Topology (Berlin: Heldermann Verlag) (1989)
[8] Kelley J L , General Topology (New York: D. Van Nostrand Reinhold Co.) (1955)
[9] O'Regan D, Fixed point theory on extension type spaces and essential maps on topological spaces, *Fixed Point Theory and Applications* **2004(1)** (2004) 13–20
[10] O'Regan D, Admissible spaces and α-fixed points, *Nonlinear Funct. Anal. Appl.* (to appear)
[11] Park S, A unified fixed point theory of multimaps on topological vector spaces, *J. Korean Math. Soc.* **35** (1998) 803–829